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On the conjugacy class graphs of some dicyclic groups

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Abstract

Let G be a dicyclic group and $\Gamma(G)$ be the attached graph related to its conjugacy classes, which is defined as: the vertices of $\Gamma(G)$ are represented by the non-central conjugacy classes of G and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. In this paper, we calculate the clique number and the girth of $\Gamma(G)$ for dicyclic groups of orders $4p, 8p, 4p^2, 4pq$ and 2^m.

Keywords: Dicyclic group, Conjugacy class, Clique number, Girth.

2020 MSC: 05C25, 20D60.

1. Introduction

There are many possible ways for associating a graph with a group, for the purpose of investigating these algebraic structures using properties of the associated graph, see for example [[1], [2], [5], [7]]. Let G be a finite group and $V(G)$ be the set of all non-central conjugacy classes of G. From orders of representatives of conjugacy classes, the following conjugacy class graph $\Gamma(G)$ was introduced in [8]: its vertex set is the set $V(G)$ and two distinct vertices x^G and y^G are connected with an edge if $(o(x), o(y)) > 1$. This graph has been widely studied. See, for instance [3] and [6]. A subset X of the vertices of Γ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and is denoted by $\omega(\Gamma)$. A graph Γ is connected if there is a path between each pair of the vertices of Γ. The length of the shortest cycle in a graph Γ is called the girth of Γ and is denoted by girth(Γ). Recall that $Dic_n = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$ is a dicyclic group of order $4n$ ($n \geqslant 2$). In this paper, we calculate the clique number and the girth of conjugacy class graph of dicycle groups of orders $4p, 8p, 4p^2, 4pq$ and 2^m, where p and q are two odd primes and m is a natural number.

2. Preliminary Results

In this section we give some preliminaries which will be used in the proof of our main result.

Lemma 2.1. [4] *The group* $G = Dic_n$ *has precisely* $(n+3)$ *conjugacy classes:*
$\{1\}, \{a^n\}, \{a^i, a^{-i}\}(1 \leqslant i \leqslant n-1), \{a^{2j}b, 0 \leqslant j \leqslant n-1\}, \{a^{2j+1}b, 0 \leqslant j \leqslant n-1\}$.

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Lemma 2.2. *[4] Let* $G = Dic_n$ *be a dicyclic group of order* $4n$ $(n \geqslant 2)$*. If* g_i *for* $1 \leqslant i \leqslant n+3$ *are the representatives of the conjugacy classes of* G*, then we have table 1:*

Table 1: Representatives of the conjugacy classes of a dicyclic group of order $4n$

g_i	1	a^n	a^r $(1 \leqslant r \leqslant n-1)$	b	ab
$o(g_i)$	1	2	$\frac{2n}{(2n,r)}$	4	4

Lemma 2.3. *Let* $G = Dic_n$ *be a dicyclic group of order* $4n$ $(n \geqslant 2)$*. If* $n = p$*, where* p *is an odd prime, then the number of conjugacy classes of* G *with representatives of type* a^r $(1 \leqslant r \leqslant n-1)$ *is given in Table 2.*

Table 2: Representatives of type a^r in dicyclic groups of order 4p

$o(a^r)$ $(1 \leqslant r \leqslant n-1)$	The number of conjugacy classes of G with representatives of type a^r
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$

Proof. Suppose that G is a dicyclic group of order $4p$. By Lemma 2.1 and Lemma 2.2, G has $p-1$ non-central conjugacy classes of size 2, ($\frac{p-1}{2}$ of these classes have representatives of orders p and the remaining $\frac{p-1}{2}$ classes have representatives of orders $2p$) and 2 non-central conjugacy classes of size p (with representatives of orders 4). So, the result follows. □

Lemma 2.4. *Let* $G = Dic_n$ *be a dicyclic group of order* $4n$ $(n \geqslant 2)$*. If* $n = p^2$*, where* p *is an odd prime, then the number of conjugacy classes of* G *with representative of type* a^r $(1 \leqslant r \leqslant n-1)$ *is listed in Table 3.*

Table 3: Representatives of type a^r in dicyclic groups of order $4p^2$

$o(a^r)$ $(1 \leqslant r \leqslant n-1)$	The number of conjugacy classes of G with representatives of type a^r
$2p^2$	$\frac{\varphi(2p^2)}{2} = \frac{p(p-1)}{2}$
p^2	$\frac{\varphi(p^2)}{2} = \frac{p(p-1)}{2}$
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$

Proof. Suppose that G is a dicyclic group of order $4p^2$. By Lemma 2.1 and Lemma 2.2, G has p^2-1 non-central conjugacy classes of size 2, ($\frac{p(p-1)}{2}$ of these classes have representatives of orders p^2, $\frac{p(p-1)}{2}$ of these classes have representatives of orders $2p^2$, $\frac{p-1}{2}$ of these classes have representatives of orders p and the remaining $\frac{p-1}{2}$ classes have representatives of orders $2p$) and 2 non-central conjugacy classes of size p^2 (with representatives of orders 4). The result is deduced by these facts. □

Lemma 2.5. *Let* $G = Dic_n$ *be a dicyclic group of order* $4n$ $(n \geqslant 2)$*. If* $n = pq$*, where* p *and* q *are distinct odd primes, then the number of conjugacy classes of* G *with representatives of type* a^r $(1 \leqslant r \leqslant n-1)$ *is listed in Table 4.*

Table 4: Representatives of type a^r in dicyclic groups of order $4pq$

$o(a^r)$ $(1 \leqslant r \leqslant n-1)$	The number of conjugacy classes of G with representatives of type a^r
$2pq$	$\frac{\varphi(2pq)}{2} = \frac{(p-1)(q-1)}{2}$
pq	$\frac{\varphi(pq)}{2} = \frac{(p-1)(q-1)}{2}$
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$
q	$\frac{\varphi(q)}{2} = \frac{q-1}{2}$
$2q$	$\frac{\varphi(2q)}{2} = \frac{q-1}{2}$

Proof. Suppose that G is a dicyclic group of order $4pq$. By Lemma 2.1 and Lemma 2.2, G has $pq-1$ non-central conjugacy classes of size 2, ($\frac{(p-1)(q-1)}{2}$ of these classes have representatives of orders $2pq$, $\frac{(p-1)(q-1)}{2}$ classes have representatives of orders pq, $\frac{p-1}{2}$ classes have representatives of orders p, $\frac{p-1}{2}$ classes have representatives of orders $2p$, $\frac{q-1}{2}$ classes have representatives of orders q and $\frac{q-1}{2}$ classes have representatives of orders $2q$) and 2 non-central conjugacy classes of size pq (with representatives of orders 4). This implies the result. □

Lemma 2.6. *Let* $G = \mathrm{Dic}_n$ *be a dicyclic group of order* $4n$ $(n \geqslant 2)$. *If* $n = 2p$, *where* p *is an odd prime, then the number of conjugacy classes of* G *with representatives of type* a^r $(1 \leqslant r \leqslant n-1)$ *is given in Table 5.*

Table 5: Representatives of type a^r in dicyclic groups of order $8p$

$o(a^r)$ $(1 \leqslant r \leqslant n-1)$	The number of conjugacy classes of G with representatives of type a^r
$2p$	$\frac{\varphi(2p)}{2} = \frac{p-1}{2}$
$4p$	$\frac{\varphi(4p)}{2} = p-1$
p	$\frac{\varphi(p)}{2} = \frac{p-1}{2}$
4	$\frac{\varphi(4)}{2} = 1$

Proof. Suppose that G is a dicyclic group of order $4p$. By Lemma 2.1 and Lemma 2.2, G has $2p-1$ non-central conjugacy classes of size 2, ($\frac{p-1}{2}$ of these classes have representatives of orders $2p$, $p-1$ classes have representatives of orders $4p$, $\frac{p-1}{2}$ classes have representatives of orders p and 1 class has representative of order 4) and 2 non-central conjugacy classes of size $2p$ (with representatives of orders 4). This completes the proof. □

We draw the conjugacy class graphs of some dicyclic groups in the following example.

Example 2.7. By Table 2 and Table 5, the conjugacy class graphs of dicyclc groups of orders 24 and 28 are given in Figure 1 and Figure 2, respectively.

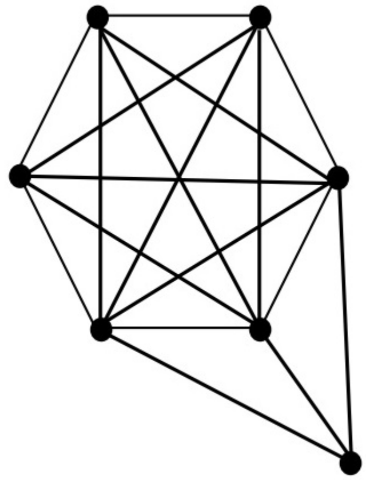

Figure 1: Conjugacy class graph of Dic_6

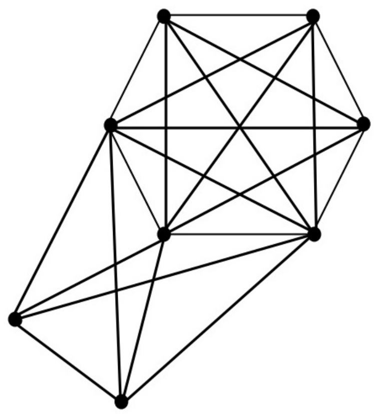

Figure 2: Conjugacy class graph of Dic_7

3. Main results

In this section, we give our main result as follows.

Theorem 3.1. *Let* $G = \mathrm{Dic}_n$ *be a dicyclic group of order* $4n$.

i) If $n = p$, *where* p *is an odd prime, then* $\omega(\Gamma(\mathrm{Dic}_p)) = p - 1$ *for* $p \geqslant 5$ *and* $\omega(\Gamma(\mathrm{Dic}_3)) = 3$. *Also,* $\mathrm{girth}(\Gamma(\mathrm{Dic}_p)) = 3$ *for* $p \geqslant 3$.

ii) If $n = p^2$, *where* p *is an odd prime, then* $\omega(\Gamma(\mathrm{Dic}_{p^2})) = p^2 - 1$ *and* $\mathrm{girth}(\Gamma(\mathrm{Dic}_{p^2})) = 3$.

iii) If $n = pq$, *where* p *and* q *are distinct odd primes, such that* $p > q$, *then* $\omega(\Gamma(\mathrm{Dic}_{pq})) = q(p-1)$ *and* $\mathrm{girth}(\Gamma(\mathrm{Dic}_{pq})) = 3$.

iv) If $n = 2p$, *where* p *is an odd prime, then* $\omega(\Gamma(\mathrm{Dic}_{2p})) = 2(p-1)$ *for* $p \geqslant 7$ *and* $\omega(\Gamma(\mathrm{Dic}_{2p})) = \frac{3p+3}{2}$ *for* $p = 3, 5$. *Also* $\mathrm{girth}(\Gamma(\mathrm{Dic}_{2p})) = 3$ *for* $p \geqslant 3$.

v) If $n = 2^m$, *where* m *is a positive integer, then* $\omega(\Gamma(\mathrm{Dic}_{2^m})) = 2^m + 1$ *and* $\mathrm{girth}(\Gamma(\mathrm{Dic}_{2^m})) = 3$.

Proof. The result follows by Lemmas 2.3-2.6 and the definition of the conjugacy class graph $\Gamma(G)$. Note that if $n = 2^m$, then $\Gamma(G)$ is the complete graph K_{2^m+1}. □

References

[1] A. Abdollahi, S. Akbari, H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, **298** (2006), 468–492. 1

[2] E. A. Bertram, *Some applications of graph theory to finite groups*, Discrete Math., **44** (1983), 31–43. 1

[3] Z. Foruzanfar, Z. Mostaghim, *Classification of non-solvable groups with a given property*, Proc. Math. Sci., **125(1)** (2015), 29–36. 1

[4] G. James, M. Liebeck, *Representations and Characters of Groups*, Cambridge University Press, 1993. 2.1, 2.2

[5] A. R. Moghaddamfar, W. J. Shi, W. Zhou, A. R. Zokayi, *On non-commuting graph associated with a finite group*, Siberian Math. J., **46** (2005), 325–332. 1

[6] M. Rezaei, Z. Foruzanfar, *Classification of solvable groups with a given property*, Bull. Iranian Math. Soc., **43(6)** (2017), 1791–1800. 1

[7] Y. Segev, *The commuting graph of minimal nonsolvable groups*, Geom. Dedicata., **88(13)** (2001), 55–66. 1

[8] X. You, G. Qian, *A new graph related to conjugacy classes of finite groups*, Chinese Ann. Math. Ser. A, **28(5)** (2007), 631–636. 1